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Characterisations of Chebyshev sets in c_0 ☆

Alexey R. Alimov*

Department of Mechanics and Mathematics, Moscow Lomonosov State University, Moscow 119992, Russia

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Abstract

A subset $M \subset X$ of a normed linear space X is a Chebyshev set if, for every $x \in X$, the set of all nearest points from M to x is a singleton. We obtain a geometrical characterisation of approximatively compact Chebyshev sets in c_0. Also, given an approximatively compact Chebyshev set M in c_0 and a coordinate affine subspace $H \subset c_0$ of finite codimension, if $M \cap H \neq \emptyset$, then $M \cap H$ is a Chebyshev set in H, where the norm on H is induced from c_0.

MSC: 41A65

Keywords: Chebyshev set; Sun; c_0

1. Introduction

A subset $\emptyset \neq M \subset X$ of a normed linear space $(X, ||\cdot||)$ is a *Chebyshev set* if, for every $x \in X$, the set $P_M x = \{y \in M \mid ||x-y|| = \rho(x, M)\}$ of its nearest points from M consists of one point. Here $\rho(x, M) = \inf_{z \in M} ||x-z||$ is the distance from x to M. The best general references here are [5,17].

The paper contains two main results. Theorem 4 characterises approximatively compact Chebyshev sets in c_0 in terms of their intersections with coordinate affine

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*Corresponding address. Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, Main Building, MSU, Vorobjovy Gory, GSP Moscow 119992, Russian Federation. Fax: + 7-095-939-2090.

E-mail address: alimov@shade.msu.ru.

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subspaces of finite codimension. Theorems 1–3 establish that an intersection of an approximatively compact Chebyshev set in c_0 with a coordinate affine subspace H of finite codimension or with a finite intersection H of coordinate half-spaces preserves its approximative properties with respect to H. Similar results for $\ell^\infty(n)$ were recently obtained by the author [2,3].

To formulate the main results of the paper the notion of a coordinate subspace will play an important role. Thus, we give the following definitions:

$\mathrm{cAff}_{\omega-k}(c_0)$ $(k\in\mathbb{Z}_+)$ will denote the set of all coordinate affine subspaces of c_0 of finite codimension k; i.e., affine subspaces H which are parallel to a correspondent face F of the unit ball, $\operatorname{codim} F = k$. In other words, $H = \{x\in c_0 \mid x_{i_1} = c_1, \dots, x_{i_k} = c_k\}$ for some fixed set of indices $i_1, \dots, i_k$ and set of constants $c_1, \dots, c_k$;

$\mathrm{cAff}_k(c_0)$ $(k\in\mathbb{N})$ will denote the set of all coordinate affine subspaces of c_0 of finite dimension k (see [3]); i.e., $\mathrm{cAff}_k(c_0)$ consists of affine subspaces of the following form: $\mathrm{lin}\{e_{i_1}, \dots, e_{i_k} \mid 1\leqslant i_1<\dots<i_k<\infty\} + x$, $x\in c_0$; here $e_1, e_2, \dots$ is the natural basis of c_0.

If $m>k$ and $H\in\mathrm{cAff}_{\omega-k}(c_0)$, then $Q\in\mathrm{cAff}_{\omega-m}(H)$ means that $Q\in\mathrm{cAff}_{\omega-m}(c_0)$ and $Q\subset H$.

Further, let $M\subset c_0$, H be a coordinate affine subspace of c_0 and Q be a hyperplane in H. Then Q is said to be *locally tangential* to M in H (we write $Q\in \underset{H}{\mathrm{LocTan}}(M)$) if there exist a point $x\in H\cap M$ and its neighbourhood $\mathcal{O}(x)$ in H such that Q is a hyperplane of support to the set $M\cap\mathcal{O}(x)$ in H. The fact that the hyperplane Q is a supporting hyperplane to the set $M\cap H$ in subspace H will be denoted by $Q\in\mathrm{Tan}_H(M)$.

A point $x\in X$ is said to be a point of approximative compactness for a set $M\subset X$ if, for every sequence $(y_n)_{n\in\mathbb{N}}\subset M$ such that $\|x-y_n\|\to\rho(x,M)$, there is a convergent subsequence $(y_{n_k})\to y\in M$. A set $M\subset X$ is *approximatively compact* (AC), if every point $x\in X$ is a point of approximative compactness for M. By $\mathrm{AC}(M)=\mathrm{AC}_X(M)$ we will denote the set of all points of approximative compactness for the set M in the space X.

Also, let us denote $\mathrm{T}(M)=\{x\in X \mid \operatorname{card} P_M x = 1\}$; i.e., the set of points from X that have a unique nearest point from M; (here the letter "T" comes from the antiquated spelling of Chebyshev as Tschebysheff). Now a set M is Chebyshev in X, if $T(M)=X$. (See [5] and [12] for density and categorical properties of $\mathrm{T}(M)$, $\mathrm{AC}(M)$ and for other characteristics of approximatively compact sets.)

The importance of coordinate subspaces for approximation theory was shown in [3] for $X=\ell^\infty(n)$. Here we obtain similar results for c_0. Theorem 1 states that for an approximatively compact Chebyshev set M in c_0 and for $H\in\mathrm{cAff}_{\omega-k}(c_0)$, $k\in\mathbb{Z}_+$, if $M\cap H\neq\emptyset$, then

$$H\subset\mathrm{T}(M\cap H),$$

i.e., every point from the subspace H has a unique nearest point from the set $(M\cap H)$. In particular, $M\cap H$ is an approximatively compact Chebyshev sun (see Section 2) in H, the norm on H being induced from ℓ^∞-norm on c_0. Theorems 2 and 3 state similar results for a finite intersection of coordinate affine half-spaces of finite codimension.

The main result of the paper is Theorem 4 where a characterisation of approximatively compact Chebyshev sets in c_0 is given.

Theorem 4. *Let $M \subset c_0$ be approximatively compact. Then M is a Chebyshev set in c_0 if and only if the following two conditions are satisfied*:

(a) *the set $M \cap H$ is connected for all $k \in \mathbb{Z}_+$ and $H \in \mathrm{cAff}_{\omega-k}(c_0)$; and*
(b) *for all $k \in \mathbb{Z}_+$, $H \in \mathrm{cAff}_{\omega-k}(c_0)$ and $Q \in \mathrm{cAff}_{\omega-(k+1)}(H)$ the condition $Q \in \mathrm{locTan}_H(M)$ implies that $Q \in \mathrm{Tan}_H(M)$ and $Q \cap M$ is a singleton.*

The similar characterisation for Chebyshev sets in $\ell^\infty(n)$ was obtained in [2,3].

The paper has the following structure. In Section 2 necessary definitions and auxiliary results are given, in Section 3 we study approximative properties of intersections of approximatively compact Chebyshev sets in c_0 with coordinate hyperplanes and layers of coordinate hyperplanes (Theorems 1–3). In Section 4 we present characterisations of approximatively compact Chebyshev sets in c_0 and in $\ell^\infty(n)$ (Theorem 4 and Theorem A).

2. Auxiliary results

As usual, if $x \in X$ and $r > 0$, then $B(x,r)$, $\mathring{B}(x,r)$ and $S(x,r)$ denote closed, open ball and sphere with centre x and radius r, respectively; to simplify notation we will also denote $B = B(0,1)$, $\mathring{B} = \mathring{B}(0,1)$, $S = S(0,1)$.

For a convex set $C \subset X$ by $\mathrm{ri}\, C$, $\mathrm{rb}\, C$, $\mathrm{cone}(y, C)$ we denote relative interior, relative boundary and conical hull of C with respect to the point y: $\mathrm{cone}(y,C) = \{\alpha c + (1-\alpha)y \mid \alpha \geqslant 0,\ c \in C\}$.

The notion of a sun, introduced by Efimov and Stechkin, proved to be important in approximation theory. Let us recall that a set $M \subset X$ is a *sun* if, for every point $x \in X \backslash M$, there exists a point $y \in P_M x$ such that $y \in P_M[(1-\lambda)y + \lambda x]$ for all $\lambda \geqslant 0$.

The following lemma establishes an important property of suns: a point not lying in a sun can be separated from it by a convex cone, namely, by the supporting cone $\mathring{K}(y,x)$, two equivalent definitions of which are given below (here $x, y \in X$, $x \neq y$) [13,17, Chapter 3]:

$$\mathring{K}(y,x) = \bigcup_{\lambda > 0} \mathring{B}(\lambda x + (1-\lambda)y, \lambda\|x-y\|), \tag{1}$$

$$\mathring{K}(y,x) = \{z \in X \mid [z,y] \cap \mathring{B}(x, \|x-y\|) \neq \emptyset\}. \tag{2}$$

Lemma A (Oschman [13], see also Vlasov [17, Chapter 3]). *A set M is a sun in X if and only if, for all $x \in X \backslash M$, there exists $y \in P_M x$ such that $\mathring{K}(y,x) \cap M = \emptyset$.*

It is a well-known fact that every Chebyshev set in a finite-dimensional normed linear space is a sun (Chebyshev sets which are suns are also called Chebyshev suns);

in infinite-dimensional spaces this is no longer true (see e.g. [8,11,17, Chapter 4]). However, under additional assumptions on a Chebyshev set M or on a space X it is possible to prove solar properties of $M \subset X$. The classical result of Vlasov [17, Theorem 4.13] establishes solar properties of boundedly compact (BC) Chebyshev sets in Banach spaces. (A ~ set M is boundedly compact if $M \cap B(x,r)$ is compact for every x and $r > 0$) Moreover, a locally compact Chebyshev set with a continuous metric projection is a sun [11,16], see also [14]. (A set M is locally compact (LC) if, for every $x \in M$, there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \cap M$ is compact.)

It is clear that (BC) $\subset$ (LC) $\cap$ (AC), (AC) $\not\subset$ (LC), (LC) $\not\subset$ (AC). We note that an approximatively compact Chebyshev set M has a continuous metric projection [17, Corollary 2.2].

It is interesting to know in which spaces X

An approximatively compact Chebyshev set is a sun. (3)

The following result (see e.g. [17, Theorem 4.18]) allows us to establish (3) in spaces which satisfy the following condition (4).

Lemma B (Brosowski and Deutsch). *Suppose that the space X satisfies the condition*

$$\forall p \in S \quad \mathring{K}(p,0) \subset \bigcup \{\mathring{K}(p,y) \mid y \in S,\ p \notin \overline{S \cap \mathring{K}(p,y)}\}. \tag{4}$$

Given a Chebyshev set $M \subset X$, suppose also that for each $x \notin M$ the restriction of the metric projection P_M to the ray $\{\lambda x + (1-\lambda)P_M x \mid \lambda \geqslant 0\}$ is continuous at x. Then M is a sun.

Amir and Deutsch [4] proved that the space $C[0,1]$ satisfies (4). Therefore, in $C[0,1]$ a Chebyshev set with a continuous metric projection is a sun; i.e., (3) is true for $X = C[0,1]$.

In the following lemma we prove that (4) is true for c_0, establishing (3) for $X = c_0$.

Proposition 1. *A Chebyshev set in c_0 with a continuous metric projection is a sun.*

Further, we will prove that *if M is an approximatively compact Chebyshev set in c_0, $H \in \mathrm{cAff}_{\omega-k}$, $k \in \mathbb{N}$, then $M \cap H$ is an approximatively compact Chebyshev sun in c_0; in particular, $M \cap H$ is a Chebyshev sun in H* (see Theorem 1 below).

Proof of Proposition 1. We will establish (4) and then apply Lemma B.

Let $p \in S$. Let us take $y = -p$. From (1) it is clear that $\mathring{K}(p,-p) = \mathring{K}(p,0)$. To prove (4) we need to check that

$$p \notin \overline{S \cap \mathring{K}(p,0)}. \tag{5}$$

Suppose the contrary. Let

$$y^{(n)} \in S, \quad y^{(n)} \to p, \quad y^{(n)} \in \mathring{K}(p,0). \tag{6}$$

Let $p = (1,\dots,1,p_{k+1},\dots)$, where $|p_j| < 1$, $j \geqslant k+1$. Then from the inclusion $y^{(n)} \in \mathring{K}(p,0) = \{z \mid z_j < 1 \text{ for all } j = 1,\dots,k\}$ it follows that $y_j^{(n)} < 1$ for $1 \leqslant j \leqslant k$.

From the convergence $y^{(n)} \to p$ it follows that $y_j^{(n)} > 0$ for all $n \geqslant n_1$ and for $1 \leqslant j \leqslant k$. Further, there is an $n_0 > n_1$ such that $||y^{(n)} - p|| < 1/8$ for all $n > n_0$. Also, there is an N_2 such that $|p_N| < 1/4$ for all $N > N_2$. Now we establish that there exists N_1 such that $|y_N^{(n)}| < 1/2$ for all $N > N_1$ and $n > n_0$. To prove that statement, assume the contrary. Then we will get

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} \leqslant |y_N^{(n)}| - |p_N| \leqslant |y_N^{(n)} - p_N| \leqslant ||y^{(n)} - p|| \leqslant \frac{1}{8},$$

a contradiction. Now for $n \geqslant n_0$ we have the following estimates for the coordinates of $y^{(n)}$: $0 < y_j^{(n)} < 1$ for $1 \leqslant j \leqslant k$, $|y_j^{(n)}| < 1/2$ for $j > N_1$. Since $y^{(n)} \in S$, for every $n \geqslant n_0$ there is a $v \in (k+1, N_1)$ such that $|y_v^{(n)}| = 1$. Then,

$$||y^{(n)} - p|| \geqslant \min_{k+1 \leqslant v < N_1} |1 - |p_v|| > C > 0, \quad n > n_0,$$

contradicting the convergence $y^{(n)} \to p$ in (6). Proposition 1 is proved. □

For a coordinate affine subspace $H \subset c_0$ and $z \in c_0$ by $\mathrm{pr}_H z$ we define the natural coordinate projection of z onto H. A natural norm $|| \cdot ||_H$ on H is induced by the ℓ^∞-norm of c_0 in the following way: (1) H is set to be a linear space by fixing an arbitrary element $\theta \in H$ as H's zero element; (2) the norm $|| \cdot ||_H$ is defined as Minkowski's functional of the convex set $B(\theta, 1) \cap H$ with respect to θ.

Given an affine subspace $H \subset c_0$, its closed ball will be denoted by $B_H(x, r)$, the open ball by $\mathring{B}_H(x, r)$, the sphere by $S_H(x, r)$ and, respectively, the open supporting cone by $\mathring{K}_H(x, y)$ (here $x, y \in H$, $x \neq y$ and $r > 0$). Under this notation it is clear that $B_H(x, r) = B(x, r) \cap H$, $\mathring{B}_H(x, r) = \mathring{B}(x, r) \cap H$, $S_H(x, r) = S(x, r) \cap H$ and $\mathring{K}_H(x, y) = \mathring{K}(x, y) \cap H$.

The following geometrical somehow unexpected result will play an essential role below.

Lemma 1. *Let M be an approximatively compact Chebyshev set in c_0 and $H \in \mathrm{cAff}_{\omega-1}(c_0)$. By H^+, H^- we denote two non-overlapping open halfspaces with boundary H. Suppose that $\mathring{B}_H(x, r) \cap M = \emptyset$ for some $x \in H$ and $r > 0$. Let $\mathring{B}_H^{\pm} = \{u \in H^{\pm} \mid \mathrm{pr}_H u \in \mathring{B}_H(x, r)\}$. Then either $\mathring{B}_H^+ \cap M = \emptyset$ or $\mathring{B}_H^- \cap M = \emptyset$.*

Proof of Lemma 1. Without loss of generality we assume that $r = 1$, $x = 0$, $0 \in H$ and that $H = \{y \mid y_1 = 0\}$. As usual, let $e^1 = (1, 0, 0, \ldots)$. Let $f \in (c_0)^*$ be a functional such that $\mathrm{Ker} f = H$ and $||f|| = 1$ (in our assumptions $f(y) = y_1$). Then $H^{\pm} = \{u \in c_0 \mid f(u) \gtrless 0\}$.

Suppose the contrary: $\mathring{B}_H(0, 1) \cap M = \emptyset$, but

$$\mathring{B}_H^+ \cap M \neq \emptyset \quad \text{and} \quad \mathring{B}_H^- \cap M \neq \emptyset. \tag{7}$$

We denote

$$\bar{\alpha} = \inf\{f(u) \mid u \in \mathring{B}_H^+ \cap M\}, \quad \underline{\alpha} = \sup\{f(u) \mid u \in \mathring{B}_H^- \cap M\}. \tag{8}$$

Let us prove that

$$\bar{\alpha} - \underline{\alpha} \geqslant 2. \tag{9}$$

Assuming that (9) is false, we set $\beta = (\bar{\alpha} + \underline{\alpha})/2$. Then it is clear that $\beta + 1 > \bar{\alpha}$, $\beta - 1 < \underline{\alpha}$. This yields that $\sup \mathring{B}(\beta e^1, 1) = \beta + 1 > \bar{\alpha}$, $\inf \mathring{B}(\beta e^1, 1) = \beta - 1 < \underline{\alpha}$, whence

$$\mathring{B}(\beta e^1, 1) \cap (M \cap \mathring{B}^+) \neq \emptyset \quad \text{and} \quad \mathring{B}(\beta e^1, 1) \cap (M \cap \mathring{B}^-) \neq \emptyset. \tag{10}$$

Clearly, $\mathring{B}_H(0,1)$ separates $\mathring{B}(0,1)$. Therefore, since $\mathring{B}_H(0,1) \subset \mathring{B}(\beta e^1, 1)$ and $\mathring{B}_H(0,1) \cap M = \emptyset$, from (10) it follows that $\mathring{B}(\beta e^1, 1) \cap M$ is not connected. But this is a contradiction, since, by Wulbert's theorem [17,18, Theorem 4.1], a Chebyshev set M with a continuous metric projection is always $\mathring{V}$-connected (i.e., $M \cap \mathring{B}(y, \rho)$ connected for any choice $y \in X$ and $\rho > 0$). Therefore, our assumption that $\bar{\alpha} - \underline{\alpha} < 2$ was false and so (9) is proved.

Now from (9) and (8) it follows that

$$\mathring{B}((\bar{\alpha} - 1)e^1, 1) \cap M = \emptyset.$$

Here we also used (7) to ensure that $\bar{\alpha} < \infty$.

Moreover, (8) yields that there is a sequence $(y^{(n)}) \subset M$ such that

$$\|(\bar{\alpha} - 1)e^1 - y^{(n)}\| \to 1 = \rho(\bar{\alpha} - 1, M).$$

Since M is approximatively compact, $(y^{(n)})$ has a subsequence converging to some $\hat{y} \in M$. Clearly, $\hat{y} \in P_M(\bar{\alpha} - 1)e^1$, $f(\hat{y}) = \hat{y}_1 = \bar{\alpha}$.

Finally, since by Proposition 1, M is a sun, from Lemma A it follows that $\mathring{K}(\hat{y}, (\bar{\alpha} - 1)e^1) \cap M = \emptyset$. Here $\mathring{K}(\hat{y}, (\bar{\alpha} - 1)e^1) = \{z \mid z_1 < \bar{\alpha},\ \varepsilon_j z_j < \varepsilon_j \hat{y}_j,\ j \in I\}$, where $\varepsilon_j = \operatorname{sign} \hat{y}_j$, $I = \{i \mid |\hat{y}_i| = 1\}$. Then, clearly, $\mathring{B}_H^- \subset \mathring{K}(\hat{y}, (\bar{\alpha} - 1)e^1)$, whence $\mathring{B}_H^- \cap M = \emptyset$, contradicting (7). Lemma 1 is proved. □

3. Intersection of Chebyshev sets with coordinate hyperplanes and layers of coordinate hyperplanes

In this section we study approximative properties of an intersection of a Chebyshev set $M \subset c_0$ with a layer of coordinate affine hyperplanes and with other convex sets C. It turns out that the intersection $M \cap C$ has "good" approximative properties if C is a layer of coordinate subspaces of finite codimension in c_0 (or finite intersection of such layers); in particular, if $C \in \mathrm{cAff}_{\omega-k}(c_0)$ is a coordinate subspace of finite codimension k. On the other hand, simple examples show that $M \cap C$ may have "bad" approximative properties if C is a subspace which is not coordinate in c_0 (see Remark 2 below).

Theorem 1. *Let M be an approximatively compact Chebyshev set in c_0 and let $H \in \mathrm{cAff}_{\omega-k}(c_0)$, $k \in \mathbb{N}$, be a coordinate subspace of finite codimension. If $M \cap H \neq \emptyset$, then $M \cap H$ is a Chebyshev set in H, which is approximatively compact.*

This result will be obtained as a corollary from a more general Theorem 2 below. The similar result is also true in $\ell^\infty(n)$ (see [3]).

From Theorem 1 and Proposition 1 we have

Corollary 1. *Let M be an approximatively compact Chebyshev set in c_0 and let $H \in \mathrm{cAff}_{\omega-k}(c_0)$, $k \in \mathbb{N}$. Then $M \cap H$ is a Chebyshev sun in c_0; in particular, $M \cap H$ is a Chebyshev sun in H.*

Let $H \in \mathrm{cAff}_{\omega-1}(c_0)$ be a coordinate hyperplane, $0 \in H$, $h \in (c_0)^*$, $\|h\| = 1$, $h|_H = 0$, $a, b \in \overline{\mathbb{R}}$, $a \leqslant b$. Then by

$$h_{a,b} = h_{a,b}(H) = \{x \in c_0 \mid a \leqslant h(x) \leqslant b\} \tag{11}$$

we denote the layer of coordinate affine hyperplanes between a and b, with respect to the h. Clearly, $h_{0,0} = H$, $h_{-\infty,\infty} = c_0$.

Theorem 2. *Let M be an approximatively compact Chebyshev set in c_0 and let $h_{a,b}$ be a layer of coordinate hyperplanes as in (11). Then, if $M \cap h_{a,b} \neq \emptyset$,*

$$h_{a,b} \subset \mathrm{T}(M \cap h_{a,b}) \cap \mathrm{AC}(M \cap h_{a,b}). \tag{12}$$

In other words, every point from the layer $h_{a,b}$ has a unique nearest point from the set $(M \cap h_{a,b})$.

Simple examples show that in general $P_M x \neq P_{(M \cap h_{a,b})} x$ for $x \in h_{a,b}$.

Proof of Theorem 2. Let $x \in h_{a,b} \backslash M$. Without loss of generality we assume that $x = 0$, $\rho(0, M \cap h_{a,b}) = 1$, $H = \{y \mid y_1 = 0\}$ and that $h(e^1) = 1$. It is clear that

$$\mathring{B}_H(0,1) \cap M = \emptyset \quad \text{and} \quad \mathring{B}(0,1) \cap (M \cap h_{a,b}) = \emptyset. \tag{13}$$

At first we will prove that $0 \in \mathrm{AC}(M \cap h_{a,b})$. Let $(y^{(n)}) \in M \cap h_{a,b}$ be a minimising sequence for 0: $\|y^{(n)}\| \to 1$.

Suppose the contrary: $0 \notin \mathrm{AC}(M \cap h_{a,b})$; i.e., $(y^{(n)})$ does not have a convergent subsequence to a point from $M \cap h_{a,b}$. This implies that (compare with (13))

$$\mathring{B}(0,1) \cap M \neq \emptyset, \tag{14}$$

for otherwise the sequence $(y^{(n)})$ would be minimising from M for 0. Since $0 \in \mathrm{AC}(M)$, this sequence has to have a convergent subsequence. Clearly, the cluster point will be in $M \cap h_{a,b}$, a contradiction with our assumption that $0 \notin \mathrm{AC}(M \cap h_{a,b})$.

Without loss of generality we assume that the intersection $\mathring{B}(0,1) \cap M$ from (14) lies in $\mathring{B}_H^- \coloneqq \{y \mid h(y) < 0\}$.

Now we can apply Lemma 1: from (13) and (14) it follows that

$$\mathring{B}_H^+ \cap M = \emptyset. \tag{15}$$

Let us denote $\alpha = \sup h(M \cap \mathring{B}_H^-)$. Then $-1 < \alpha \leqslant a \leqslant 0$. Let us fix $\hat{x} = (1+\alpha)e^1$ and consider the ball $\mathring{B} + \hat{x} = \mathring{B}(\hat{x}, 1)$. It is clear that $\mathring{B}(\hat{x}, 1) \subset \mathring{B}_H^+$, therefore from (15) we conclude that $\mathring{B}(\hat{x}, 1) \cap M = \emptyset$. Let us prove that

$$||\hat{x} - y^{(n)}|| \to 1, \tag{16}$$

i.e., that $y^{(n)}$ is a minimising sequence from $M \cap h_{a,b}$ for $\hat{x}$.

Since $||y^{(n)}|| \to 1$, for any $\varepsilon > 0$ there is an $N > 0$ such that $1 \leqslant ||y^{(n)}|| < 1 + \varepsilon$ for every $n > N$. Further, $y^{(n)} \in h_{a,b}$ implies that

$$|y_j^{(n)}| < 1 + \varepsilon \quad \text{for every } j \geqslant 2 \text{ and } a \leqslant y_1^{(n)} < 1 + \varepsilon. \tag{17}$$

Therefore,

$$||\hat{x} - y^{(n)}|| = ||y^{(n)} - (1+\alpha)e_1|| = \max\left\{ |y_1^{(n)} - (1+\alpha)|, \sup_{j \geqslant 2} |y_j^{(n)}| \right\} < 1 + \varepsilon,$$

where the last inequality follows from (17) and the inequalities $-1 < \alpha \leqslant a \leqslant 0$. This is our claim in (16). Now $y^{(n)}$ is a minimising sequence from M for $\hat{x}$. Since $M \in (\mathrm{AC})$, it follows that $y^{(n)}$ has a convergent subsequence. Clearly, its cluster point lies in $M \cap h_{a,b}$. This contradicts our assumption that $y^{(n)}$ does not have subsequences converging to points from $M \cap h_{a,b}$. Thus we have proved the first half of (12): $h_{a,b} \subset \mathrm{AC}(M \cap h_{a,b})$.

The fact that $h_{a,b} \subset T(M \cap h_{a,b})$ will be proved in the same way. Supposing that $y', y'' \in P_{(M \cap h_{a,b})}0$, $y' \neq y''$, we similarly have that $y', y'' \in P_{(M \cap h_{a,b})}\hat{x}$ for $\hat{x}$ defined in above. Now $\mathring{B}(\hat{x}, 1) \cap M = \emptyset$ and, therefore, $y', y'' \in P_M \hat{x}$. This contradicts our assumption on M to be a Chebyshev set. Theorem 2 is proved. □

It is natural to consider an intersection of a finite number of coordinate layers of the form (11):

$$G = \bigcap_{k=1}^{n} h_{a_k, b_k}(H_k), \tag{18}$$

where $H_k \in \mathrm{cAff}_{\omega-1}(c_0)$, $a_k, b_k \in \overline{\mathbb{R}}$, $k = 1, \dots, n$. We will call such an intersection a *coordinate box*. Similar arguments as in the proof of Theorem 2 give the following generalisation of Theorem 2 for coordinate boxes in c_0.

Theorem 3. *Let M be an approximatively compact Chebyshev set in* c_0 *and let G be a coordinate box of the form* (18). *Then, if* $M \cap G \neq \emptyset$,

$$G \subset \mathrm{T}(M \cap G) \cap \mathrm{AC}(M \cap G). \tag{19}$$

In other words, every point from the box G has a unique nearest point from the set $(M \cap G)$.

Remark 1. Theorem 1 says that *an intersection of a approximatively compact Chebyhsev set $M \subset c_0$ with a coordinate affine subspace $P \in \mathrm{cAff}_{\omega-k}(c_0)$ of finite codimension is an approximatively compact Chebyshev set in P*. It is unknown for the author whether the same remains true if $\operatorname{codim} P = \infty$. (The answer is positive for any P with $\dim P = 1$ which is easy to verify.)

Remark 2. The geometric form of the sets (11) and (18) is important. *Sets $h_{a,b}$ and G cannot be replaced by arbitrary convex sets*. It is easy to construct a Chebyshev set $M \subset \ell^\infty(2)$ and a convex set G such that $M \cap G$ is not acyclic (therefore, M is never a Chebyshev set for *any* norm on G) (i.e., $G \not\subset T(M \cap G)$). Let us consider $M = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 1,\ x_1 > 0\}$ and a non-coordinate line $G = \{x_1 + x_1 = 3\}$. Then $G \cap M$ is disconnected and therefore is never a Chebyshev set in G for any norm or non-symmetric norm on G.

4. Characterisation of Chebyshev sets in c_0 and $\ell^\infty(n)$

At first we note that Chebyshev sets in $\ell^\infty(n)$ and approximatively compact Chebyshev sets in $C(Q)$ with compact metrisable Q were characterised by Dunham [10] using the properties of regularity and zero-sign compatibility.

In [2,3] we presented the following geometrical characterisation of Chebyshev sets in $\ell^\infty(n)$.

Theorem A. *A set M is a Chebyshev set in $\ell^\infty(n)$ if and only if the following three conditions are satisfied*:

(a) *M is closed*;
(b) *$M \cap H$ is connected for every $k = 1, \ldots, n$ and $H \in \mathrm{cAff}_k(\mathbb{R}^n)$*;
(c) *for every $k = 2, \ldots, n$, $H \in \mathrm{cAff}_k(\mathbb{R}^n)$ and $Q \in \mathrm{cAff}_{k-1}(H)$ the condition that Q is a locally supporting hyperplane for M in H implies that $Q \in \mathrm{Tan}_H(M)$ and $Q \cap M$ is a singleton.*

In particular, this result gave a positive answer to the question of whether the intersection of a Chebyshev set in $\ell^\infty(n)$ with a coordinate subspace H will be a Chebyshev set in H.

Let us recall that suns in $\ell^\infty(n)$ were characterized by Berens and Hetzelt [6], see also Brown [9]; strict suns in $\ell^\infty(n)$ were geometrically described by Brosowski [7] and the author [1].

Now we give a similar characterisation for approximatively compact Chebyshev sets in c_0.

Theorem 4. *Let $M \subset c_0$ be approximatively compact. Then M is a Chebyshev set in c_0 if and only if the following two conditions are satisfied*:

(a) *the set $M \cap H$ is connected for all $k \in \mathbb{Z}_+$ and $H \in \mathrm{cAff}_{\omega-k}(c_0)$; and*
(b) *for all $k \in \mathbb{Z}_+$, $H \in \mathrm{cAff}_{\omega-k}(c_0)$ and $Q \in \mathrm{cAff}_{\omega-(k+1)}(H)$ the condition $Q \in \mathrm{locTan}_H(M)$ implies that $Q \in \mathrm{Tan}_H(M)$ and $Q \cap M$ is a singleton.*

Proof of Theorem 4. The "ONLY IF" part. Let $M \subset c_0$ be an approximatively compact Chebyshev set and let $H \in \mathrm{cAff}_{\omega-k}(c_0)$ be such that $\emptyset \neq M \cap H$. Theorem 2 shows that $M \cap H$ is an approximatively compact Chebyshev set in H. Therefore, the metric projection $P_M : H \to M \cap H$ is continuous [17, Corollary 2.2]. Now the connectedness of $M \cap H$ follows from a classical result [18] that a Chebyshev set with a continuous metric projection is $\mathring{V}$-connected, and, therefore, arcwise-connected; see [17,11, Theorem 4.1]).

Let $k \in \mathbb{Z}_+$, $H \in \mathrm{cAff}_{\omega-k}(c_0)$ and let $Q \in \mathrm{cAff}_{\omega-(k+1)}(H)$, $Q \in \mathrm{locTan}_H(M)$. Without loss of generality we assume that $0 \in Q$. Let $x \in Q \cap M$ and let $\mathcal{O}(x)$ be a convex neighbourhood of x in H such that Q is a hyperplane of support to the set $M \cap \mathcal{O}(x)$ at x in H. Then $\mathring{B}(x, r) \subset \mathcal{O}(x)$ for some $r > 0$. Let φ be a continuous linear functional on H with $\mathrm{Ker}\, \varphi = Q$. Let us denote $Q^- = \{z \in H \mid \varphi(z) < 0\}$. Changing if necessary φ with $-\varphi$ we have $Q^- \cap (M \cap \mathcal{O}(x)) = \emptyset$.

Let us fix a point $\xi \in Q^-$ such that $\mathrm{pr}_Q \xi = x$ and $||x - \xi|| < r/2$. Then $\mathring{B}_H(\xi, ||\xi - x||) \subset (Q^- \cap \mathcal{O}(x))$. By Theorem 2 $M \cap H$ is a Chebyshev set in H and by Corollary 1 $M \cap H$ is a sun. From Lemma A it follows that $\mathring{K}_H(x, \xi) \cap M = \emptyset$. Since $\mathrm{pr}_H \xi = x$, we see that x is a point of smoothness of the ball $B_H(\xi, ||\xi - x||)$, whence $\mathring{K}_H(x, \xi)$ is an open half-space that is equal to Q^-. Therefore $\mathring{K}_H(x, \xi) \cap M = \emptyset$ and we have proved that Q is a supporting hyperplane to the set $M \cap H$ at x.

To prove the uniqueness of the intersection $M \cap Q$, recall that

$$\mathring{K}_H(x, \xi) = \bigcup_{\alpha > 0} \mathring{B}_H(\alpha\xi + (1 - \alpha)x, \alpha||x - \xi||), \tag{20}$$

whence x is a unique nearest point from $M \cap H$ to every point $\alpha\xi + (1 - \alpha)x$, where $\alpha > 0$. Further, since $\mathrm{bd}\, \mathring{K}_H(x, \xi) = Q$ and $M \cap H$ is a Chebyshev set, we finally obtain that $M \cap Q = \{x\}$. Thus (b) is fulfilled.

The "IF" part. For an approximatively compact set M in c_0, having assumed that (a) and (b) are fulfilled, let us prove that M is a Chebyshev set in c_0.

Let $x \notin M$. We will prove that x has a unique nearest point from M. Without loss of generality we put $x = 0$, $\rho(0, M) = 1$.

Since M is approximatively compact, $P_M z \neq \emptyset$ for every $z \in c_0$ (see [17, Proposition 2.2]). Let $y \in P_M 0$. The fact that all the faces of the unit ball of c_0 are faces of finite codimension will play a key role in our proof.

For $z \in S(0, 1)$ let $E(z)$ denote (a unique) face of B such that $z \in \mathrm{ri}\, E(z)$. It is easy to see that $E(z) = \{\varepsilon = (\varepsilon^k) \mid \varepsilon^k = 1,\ \text{if}\ z^k = 1;\ \varepsilon^k = -1,\ \text{if}\ z^k = -1;\ \varepsilon^k \in [-1, 1],\ \text{if}\ |z^k| < 1\}$. Let us also denote $\mathcal{F}(z) = \{F \mid F$ is a proper face of B such that $z \in F$ and $z \notin \mathrm{ri}\, F\}$.

Let us consider the convex body $K(y,0) = \operatorname{cl} \mathring{K}(y,0)$. From (20) it follows that the only faces that $K(y,0)$ has are the conical hulls $\operatorname{cone}(y,F)$, where either $F \in \mathcal{F}(y)$ or $F = E(y)$. Theorem 18.2 from [15] states that for a convex body C

$$\operatorname{rb} C = \cup\{\operatorname{ri} F \mid F \text{ is a proper face of } C\}. \tag{21}$$

It follows that

$$\operatorname{bd} \mathring{K}(y,0) = \operatorname{cone}(y, E(y)) \cup \bigcup_{F \in \mathcal{F}(y)} \operatorname{cone}(y, \operatorname{ri} F). \tag{22}$$

Let $y \in P_M 0$ be such the magnitude

$$d = \min\{\operatorname{codim} E(z) \mid z \in P_M 0\} \tag{23}$$

is minimal. This definition implies that if F is a face of B and $\operatorname{codim} F < d$, then $\operatorname{ri} F \cap M = \emptyset$.

Let us prove that

$$P_M 0 = \{y\}. \tag{24}$$

1. Let $d = 1$. Fix $H = c_0$, $Q = \operatorname{aff} E(y)$. Since $\mathring{B} \cap M = \emptyset$ and $y \in \operatorname{ri} E(y)$, then Q is locally tangential hyperplane to the set M at the point y. By condition (b), Q is (globally) tangential hyperplane to the set M at x and $Q \cap M = \{x\}$. This clearly implies that $P_M 0 = \{y\}$. In the case $d = 1$ the statement (24) is proved.

2. Let $d > 1$. Without loss of generality we assume $y = (1, 1, \ldots, 1, \xi_{d+1}, \ldots)$, $|\xi_i| < 1$, $i \geqslant d+1$. Then $\mathring{K}(y,0) = \{z \mid z_1 < 1, \ldots, z_d < 1\}$.

Let us show that

$$\operatorname{aff} E(y) \cap M = \{y\}. \tag{25}$$

To prove (25) fix $\Phi \in \mathcal{F}(y)$ with $\operatorname{codim} \Phi = d-1$ (it is clear that such a face Φ exists) and then apply condition (b) to the pair $H = \operatorname{aff} \Phi \in \operatorname{cAff}_{\omega-d+1}(c_0)$, $Q = \operatorname{aff} E(y) \in \operatorname{cAff}_{\omega-d}(c_0)$ at point y.

Since by (23) $\operatorname{ri} \Phi \cap M = \emptyset$ and $y \in \operatorname{ri} E(y) \subset \operatorname{rb} \Phi$, we have

$$\operatorname{cone}(\operatorname{ri} \Phi, y) \cap M = \emptyset \quad \text{and} \quad \operatorname{aff} E(y) \cap M = \{y\}, \tag{26}$$

which proves (25).

Further, by induction on codimension $j = 1, \ldots, d-1$ of the face $F \in \mathcal{F}(y) \setminus \{E(y)\}$ let us prove that

$$\operatorname{cone}(\operatorname{ri} F, y) \cap M = \emptyset, \quad \text{and} \quad \operatorname{cone}(F, y) \cap M = \{y\} \tag{$\mathrm{i_j}$}$$

is true for every $j = d-1, \ldots, 1$.

The statement (i_{d-1}) is proved in (26). Suppose that (i_j) is true for every $j = d-2, \ldots, v+1$. We need to establish (i_v). Fix $F \in \mathcal{F}(y)$, $\operatorname{codim} F = v$.

Without loss of generality we take

$$F = \{(1, \ldots, 1, \eta_{v+1}, \ldots)\}, \quad |\eta_\mu| \leqslant 1, \quad \mu \geqslant v+1.$$

Then $\operatorname{cone}(\operatorname{ri} F, y) = \{(1, \ldots, 1, \alpha\eta_{v+1} + 1 - \alpha, \ldots, \alpha\eta_{d+1} + (1-\alpha)\xi_{d+1}, \ldots)\}$, $\alpha \geqslant 0$. Now let $G_1, \ldots, G_N$ be all $(v+1)$-codimensional faces from $\mathcal{F}(y)$. It is clear that $G_\mu \subset \operatorname{rb} F$ and $E(y) \subset G_\mu \cap F$, $\mu = 1, \ldots, N$. From (22), (21) and from the structure of

G_μ it follows that

$$\operatorname{rb}\operatorname{cone}(\operatorname{ri} F, y) = \bigcup_{\mu=1,\dots,N} \operatorname{cone}(G_\mu, y) \tag{27}$$

and, further, from (27) and (i_{v+1}) we have

$$\operatorname{rb}\operatorname{cone}(\operatorname{ri} F, y) \cap M = \{y\}. \tag{28}$$

Since $v < d-1$, from (23) it follows that $\operatorname{ri} F \cap M = \emptyset$. Now, from the connectedness of the intersection $M \cap \operatorname{aff} F$ (condition (a)) and from the fact that $\operatorname{ri} F \cap M = \emptyset$, applying (28) we have:

$$\operatorname{cone}(\operatorname{ri} F, y) \cap M = \emptyset. \tag{29}$$

Finally, from (28) and (29) we get (i_v).

Thus, (i_j) is fulfilled for every $j = 1, \dots, d-1$.

Now (22), (25) and $(\mathrm{i}_1), \dots, (\mathrm{i}_{d-1})$ imply that

$$\operatorname{bd} \mathring{K}(y, 0) \cap M = \{y\}. \tag{30}$$

Applying condition (a) to $H = c_0$ and using that $\mathring{B} \cap M = \emptyset$, from (30) we finally obtain that

$$\mathring{K}(y, 0) \cap M = \emptyset \quad \text{and} \quad \operatorname{bd} \mathring{K}(y, 0) \cap M = \{y\}.$$

This shows that $P_M 0 = \{y\}$. Theorem 4 is proved. □

Example. Let $a = (1, 1/2, 1/3, \dots)$, $b = (-1/2, -1/3, -1/4, \dots)$. Then the line segment $M = [0, a]$ and the non-convex union $[0, a] \cup [0, b]$ of two-line segments serve as examples of boundedly compact Chebyshev sets in c_0.

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